Cyril Bachelard
OLZ AG

Optimal Portfolios and Where to Find Them

Hint: Somewhere inside a High-Dimensional Convex Polytope

Agenda

1. **Introduction**
2. Dear Investor, What's Your Objective?
3. Dear Audience, Remember Geometry?
4. Dear Manager, Are You Reliable Enough?
5. Dear Solver, Can You Find It?

1. About me

MSc in Economics *University of Bern*

Research and Product Development (2011-now)
OLZ AG

OLZ AG was founded in 2001 by

- C. **O**rlacchio
- Prof. C. **L**oderer (*University of Bern*)
- P. **Z**graggen

Headquartered in **Bern** with offices and subsidiaries in

- **Zurich**
- **Liechtenstein**
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The investment philosophy is

Efficient Investing

asset management without conflicts of interest
scientifically sound investment concept

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The main focus is on

Risk Based Strategies

OLZ AG provides investment solutions for

- **Institutional Investors**
- **Private Clients**

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The **objective function** f can be

Linear and Quadratic
Ex.: Mean-Variance (Markowitz)

$$f(\mathbf{w}) = \boldsymbol{\mu}^T \mathbf{w} - \frac{\lambda}{2} \mathbf{w}^T \Sigma \mathbf{w}$$

where

$$\boldsymbol{\mu} = \mathbb{E}[\mathbf{X}]$$
$$\Sigma = \mathbb{E}\left[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T\right]$$

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Non-linear
Ex.: Prospect Theory (Kahneman-Tversky)

$$f(\mathbf{w}) = \mathbb{E}\left[g\left(\mathbf{w}^T \mathbf{X}\right)\right]$$

where

$$g(x) = \begin{cases} (x-\theta)^a & x \geq \theta \\ -b(-(x-\theta))^a & x < \theta \end{cases}$$

$a \in [0,1], b > 1.$

The **constraints** $\mathcal{C}$ can be

Budget	$\mathbf{1}^T\mathbf{w} = 1$
No Short-Selling (Lower Bounds) **Liquidity** (Upper Bounds)	$\mathbf{b}_l \leq \mathbf{w} \leq \mathbf{b}_u$
Country **Sector** **Asset Classes**	$A\mathbf{w} \leq \mathbf{b}$, with A binary

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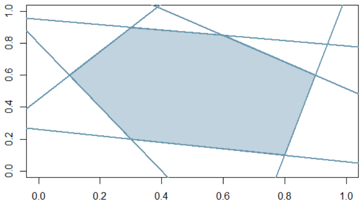
Tracking Error **Variance**	$(\mathbf{w} - \bar{\mathbf{w}})^T \, \Sigma \, (\mathbf{w} - \bar{\mathbf{w}}) \leq \bar{c}$
Turnover	$\lVert \mathbf{w} - \bar{\mathbf{w}} \rVert_1 \leq \bar{c}$
Tail Risk	$\mathrm{g}(\mathbf{w}) \leq 0$

The **constraints** $\mathcal{C}$ can be

Linear: $A\mathbf{w} \leq \mathbf{b}$



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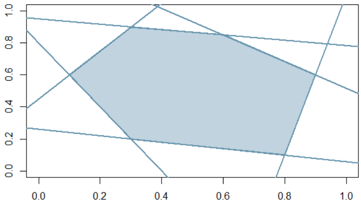
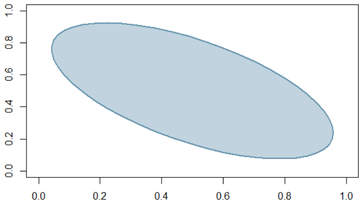


Quadratic: $\mathbf{w}^T Q\mathbf{w} \leq r$



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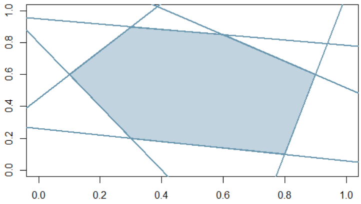

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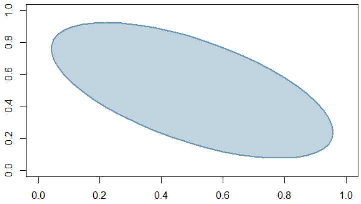
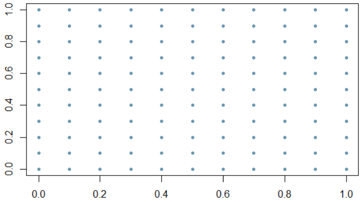


Mixed Integer: $\mathbf{w} \in \mathbb{Z}^n$



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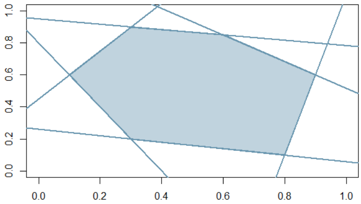

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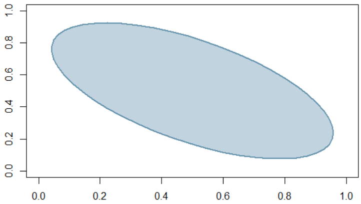

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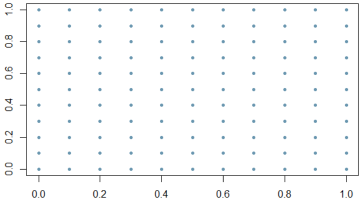
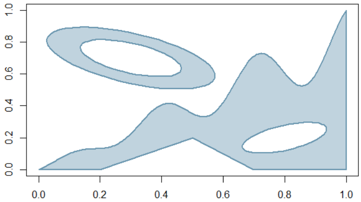


Non-Linear: $g(\mathbf{w}) \leq 0$



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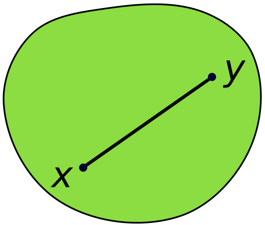
Definition (Convex Set)

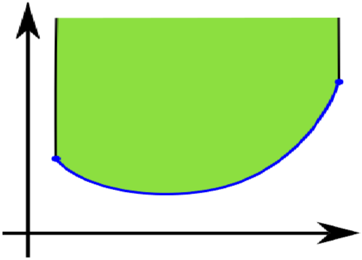
A set $C \subseteq \mathbb{R}^n$ is convex if for all $\boldsymbol{x}, \boldsymbol{y} \in C$ and for all $\alpha \in [0, 1]$, $\alpha \boldsymbol{x} + (1 - \alpha) \boldsymbol{y} \in C$.

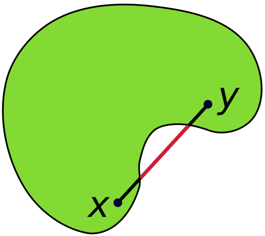


Definition (Convex Function)

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $\boldsymbol{x}, \boldsymbol{y} \in C$ and for all $\alpha \in [0, 1]$,

$$f(\alpha \boldsymbol{x} + (1 - \alpha) \boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1 - \alpha) f(\boldsymbol{y}) .$$





Definition (L^p-norm)

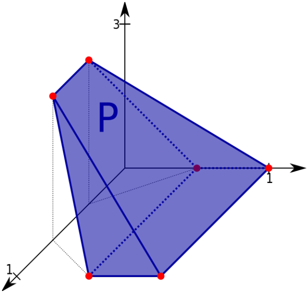
For $p \in \mathbb{R}$, $p \geq 1$, the L^p-norm of $w \in \mathbb{R}^n$ is $\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$.

Definition (Polyhedron)

An n-dimensional (convex) polyhedron is the intersection of I n-dimensional half-spaces $S_i = \left\{\mathbf{w} \in \mathbb{R}^n \,\middle|\, \mathbf{a}_i^T \mathbf{w} \leq b_i \right\}$ for $i = 1, \dots, I$

$$P = \bigcap_{i=1}^{I} S_i = \{\mathbf{w} \in \mathbb{R}^n \,|A\mathbf{w} \leq \mathbf{b}\}$$

where the i-th row of A is $\mathbf{a}_i$ for $i = 1, \dots, I$.
This is the H-representation of a polyhedron.



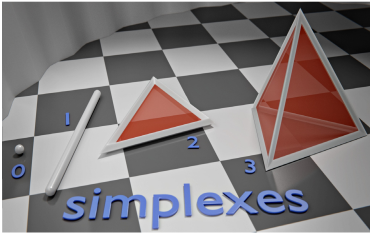
Definition (Polytope)

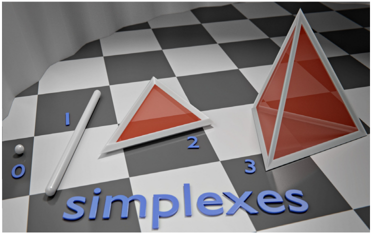
A (convex) polytope $\mathcal{P}$ is a bounded (convex) polyhedron.

Definition (Standard Simplex)

A standard n-simplex is

$$\mathcal{S} = \left\{ \boldsymbol{w} \in \mathbb{R}^{n+1} \,\middle|\, \boldsymbol{1}^T \boldsymbol{w} = 1, \boldsymbol{w} \geq 0 \right\}$$





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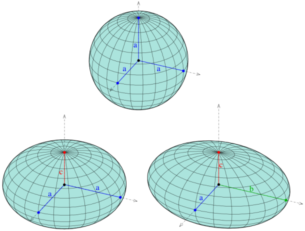

Definition (Ellopsoid)

*An ellipsoid centred in **c** with shape matrix $\Sigma \succeq 0$ is*

$$\mathcal{E}_{\Sigma,\boldsymbol{c}} = \left\{ \boldsymbol{w} \in \mathbb{R}^{n} \,\middle|\, (\boldsymbol{w} - \boldsymbol{c})^T \, \Sigma \, (\boldsymbol{w} - \boldsymbol{c}) \leq 1 \right\}$$

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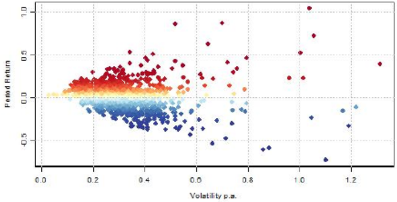
"THE PROCESS OF SELECTING a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. This paper is concerned with the second stage."

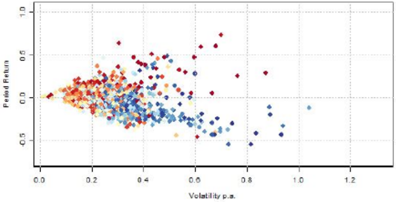
H. Markowitz (1952). *Portfolio Selection.* Journal of Finance Vol.7, No.1., pp.77-91.

Stocks sorted by return

Calculation period:
31.12.2015 – 30.03.2016

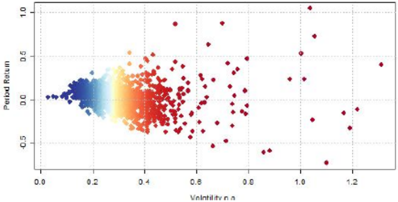


Subsequent period:
31.03.2016 – 28.06.2016



Stocks sorted by risk

Calculation period:
31.12.2015 – 30.03.2016



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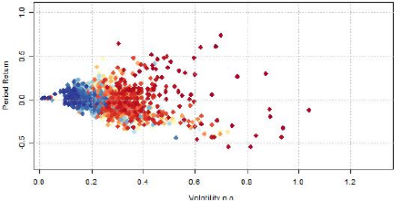

Source: Bloomberg, data MSCI World Index, own calculation.

Bias-Variance Tradeoff

The total squared error of an estimator can be decomposed as

$$\mathbb{E}\left[\left(\hat{\Sigma}-\Sigma\right)^2\right] = \mathrm{Var}\left[\hat{\Sigma}\right] + \mathrm{Bias}\left[\hat{\Sigma},\Sigma\right]^2$$

Variance

An estimator $\hat{\Sigma}$ computed on real data of the quantity Σ is affected by noise.

$\mathrm{Var}\left[\hat{\Sigma}\right] = \mathbb{E}\left[\hat{\Sigma}^2\right] - \mathbb{E}\left[\hat{\Sigma}\right]^2$ is high.

Bias

A matrix $\bar{\Sigma}$ chosen for it's structure has a bias w.r.t. Σ.

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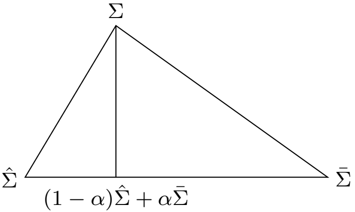

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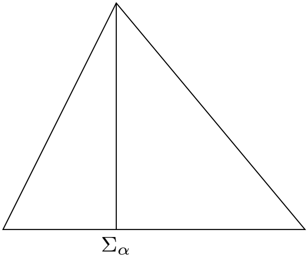
Shrinkage

Σ	Real Covariance Matrix
$\hat{\Sigma}$	Estimated Covariance Matrix
$\bar{\Sigma}$	Shrinkage Target
α	Shrinkage Intensity

$$\Sigma = \begin{bmatrix} 1.06 & -0.21 & 0.78 \\ -0.21 & 0.95 & -0.28 \\ 0.78 & -0.28 & 1.1 \end{bmatrix}$$



$$\begin{bmatrix} 1.08 & -0.2 & 0.8 \\ -0.2 & 0.92 & -0.3 \\ 0.8 & -0.3 & 1.12 \end{bmatrix} = \hat{\Sigma}$$

$$\bar{\Sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Sigma_\alpha = (1-\alpha)\hat{\Sigma} + \alpha\bar{\Sigma} = \begin{bmatrix} 1.06 & -0.2 & 0.8 \\ -0.2 & 0.94 & -0.3 \\ 0.8 & -0.3 & 1.09 \end{bmatrix} \text{ with } \alpha = 0.25$$

Theorem (Shrinkage Interpretation of Constraints)

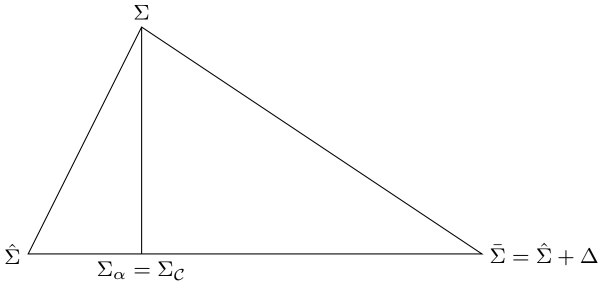
Given a covariance matrix $\hat{\Sigma}$ and a set of constraints $\mathcal{C}$ there exist $\Sigma_{\mathcal{C}}$ such that

$$\operatorname*{argmin}_{\boldsymbol{w}\in\mathcal{C}} \boldsymbol{w}^T \hat{\Sigma} \boldsymbol{w} = \hat{\boldsymbol{w}} = \operatorname*{argmin}_{\boldsymbol{w}\in\mathbb{R}^n} \boldsymbol{w}^T \Sigma_{\mathcal{C}} \boldsymbol{w}$$

In addition

$$\Sigma_{\mathcal{C}} = \hat{\Sigma} + \alpha\Delta = (1-\alpha)\,\hat{\Sigma} + \alpha\left(\hat{\Sigma} + \Delta\right)$$

so $\Sigma_{\mathcal{C}}$ is a shrinked version of $\hat{\Sigma}$.

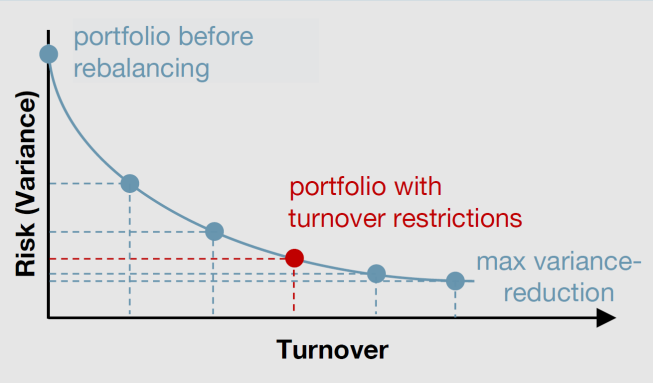


Variance-Turnover Tradeoff

When reallocating a portfolio, it can be interesting to have a turnover constraint in order to limit the impact of transaction costs

$$\|\mathbf{w} - \bar{\mathbf{w}}\|_1 \leq \bar{c}.$$

The marginal benefit (risk reduction) decreases substantially when turnover increases.



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Objective Function

Constraints	Linear	Quadratic	Non-Linear
Linear	**LP**	**QP**	**NLP**
Quadratic	**QCLP**	**QCQP**	**NLP**
Mixed-Integers	**MILP**	**MIQP**	**MINLP**
Non-Linear	**NLP**	**NLP**	**GNLP**

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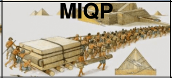
Mixed-Integers	 **MILP**	 **MIQP**	 **MINLP**
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Objective Function

Constraints

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Quadratic	 QCLP	 QCQP	 NLP
Mixed-Integers	 MILP	 MIQP	 MINLP
Non-Linear	 NLP	 NLP	

References

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